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Model Checking for Probabilistic Hybrid Systems

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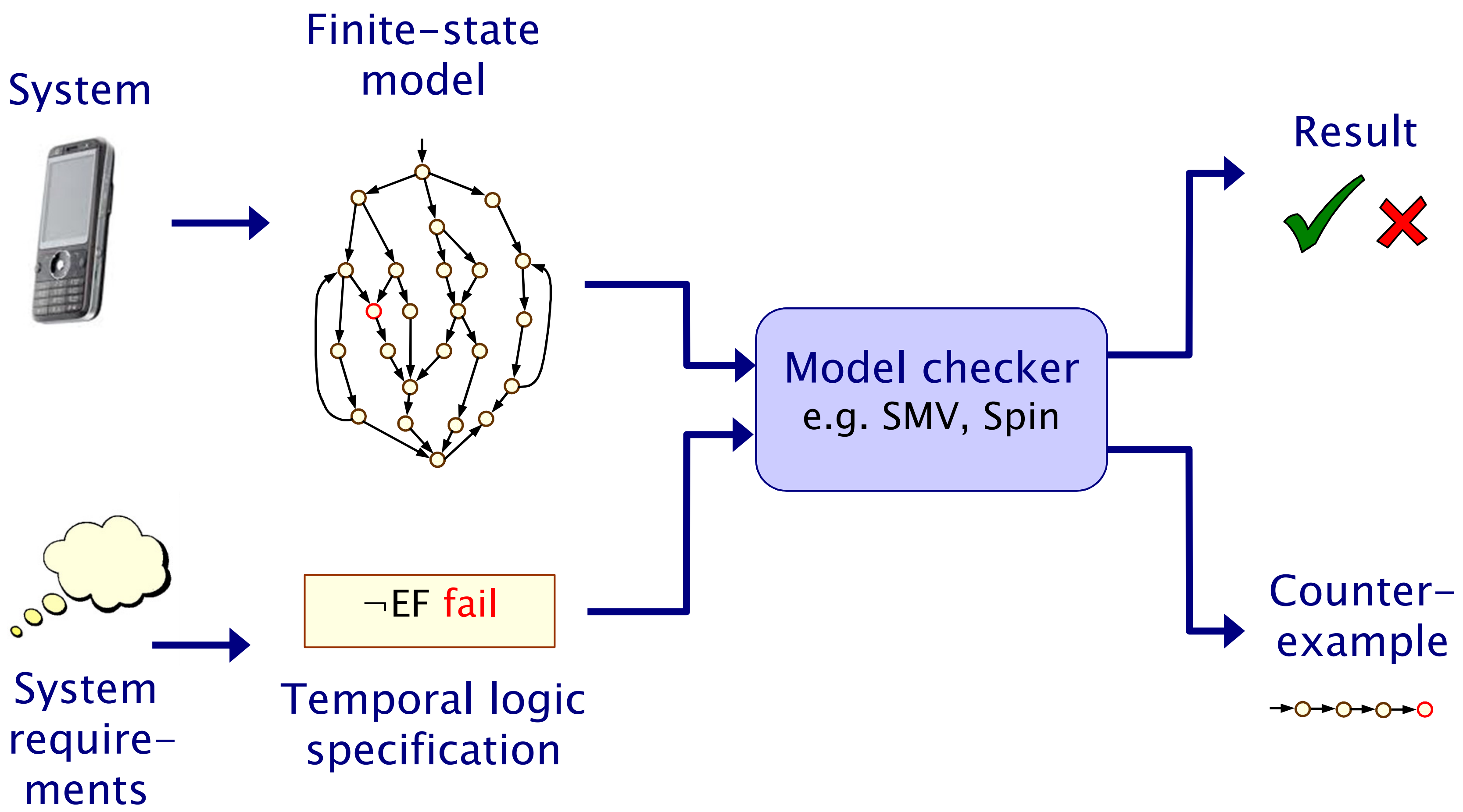
Introduction

Probabilistic models and
probabilistic model checking

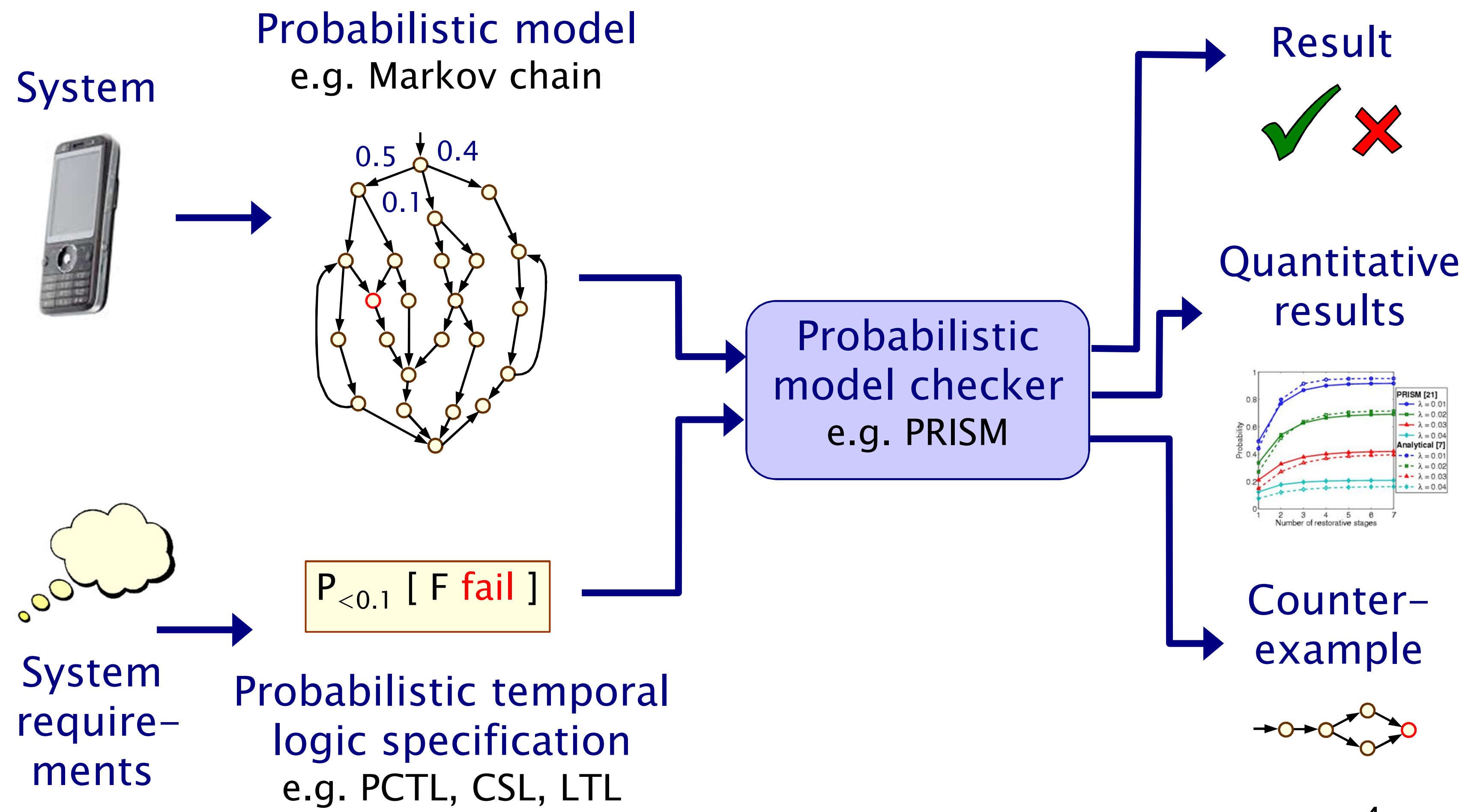
Model checking

Automated formal verification for finite-state models



Probabilistic model checking

Automatic verification of systems with probabilistic behaviour



Why probability?

- Some systems are inherently probabilistic…

- Randomisation, e.g. in distributed coordination algorithms
  - as a symmetry breaker, in gossip routing to reduce flooding

- Examples: real-world protocols featuring randomisation:
  - Randomised back-off schemes
    - CSMA protocol, 802.11 Wireless LAN
  - Random choice of waiting time
    - IEEE1394 Firewire (root contention), Bluetooth (device discovery)
  - Random choice over a set of possible addresses
    - IPv4 Zeroconf dynamic configuration (link-local addressing)
  - Randomised algorithms for anonymity, contract signing, …

Why probability?

- Some systems are inherently probabilistic…
- Randomisation, e.g. in distributed coordination algorithms
  - as a symmetry breaker, in gossip routing to reduce flooding
- To model uncertainty and performance
  - to quantify rate of failures, express Quality of Service
- Examples:
  - computer networks, embedded systems
  - power management policies
  - nano-scale circuitry: reliability through defect-tolerance

Why probability?

- Some systems are inherently probabilistic…
- Randomisation, e.g. in distributed coordination algorithms
  – as a symmetry breaker, in gossip routing to reduce flooding
- To model uncertainty and performance
  – to quantify rate of failures, express Quality of Service
- To model biological processes
  – reactions occurring between large numbers of molecules are naturally modelled in a stochastic fashion

Verifying probabilistic systems

- We are not just interested in correctness
- We want to be able to quantify:
  - security, privacy, trust, anonymity, fairness
  - safety, reliability, performance, dependability
  - resource usage, e.g. battery life
  - and much more…
- Quantitative, as well as qualitative requirements:
  - how reliable is my car's Bluetooth network?
  - how efficient is my phone's power management policy?
  - is my bank's web-service secure?
  - what is the expected long-run percentage of protein X?

Probabilistic models

- Markov Decision Process (MDP)
  - probabilistic and nondeterministic behaviour
  - already allow to express relevant class of models
  - semantic base for extended models below

- Probabilistic Timed Automata (PTA)
  - extend MDPs with clocks to express timed behaviour

- Probabilistic Hybrid Automata (PHA)
  - extend clocks of PTAs to more general continuous variables
  - often described by differential equations

Nondeterminism

- Some aspects of a system may not be probabilistic and should not be modelled probabilistically; for example:

- Concurrency - scheduling of parallel components
  - e.g. randomised distributed algorithms - multiple probabilistic processes operating asynchronously

- Underspecification - unknown model parameters
  - e.g. a probabilistic communication protocol designed for message propagation delays of between d_{min} and d_{max}

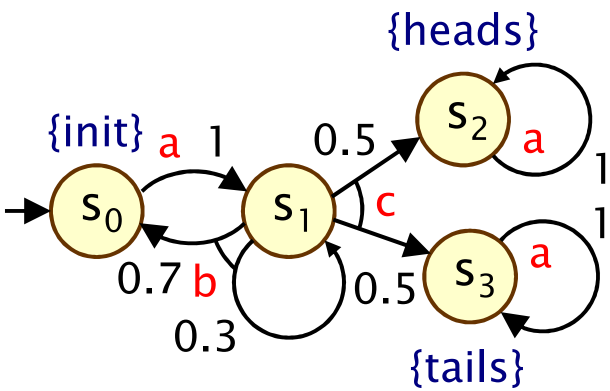
- Unknown environments
  - e.g. probabilistic security protocols - unknown adversary

Markov decision processes

- Formally, an MDP M is a tuple $(S, s_{init}, \mathbf{Steps}, L)$ where:
  - S is a finite set of states ("state space")
  - $s_{init} \in S$ is the initial state
  - $\mathbf{Steps} : S \rightarrow 2^{Act \times Dist(S)}$ is the transition probability function

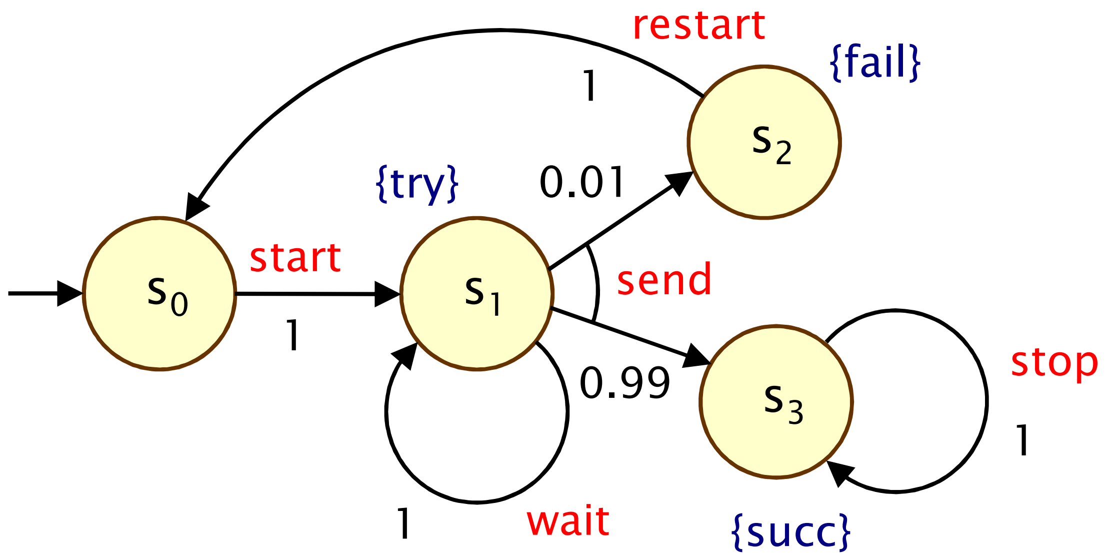
    where Act is a set of actions and Dist(S) is the set of discrete probability distributions over the set S
  - $L : S \rightarrow 2^{AP}$ is a labelling with atomic propositions

- Notes:
  - **Steps**(s) is always non-empty, i.e. no deadlocks
  - the use of actions to label distributions is optional



Simple MDP example

- Simple communication protocol
  – after one step, process starts trying to send a message
  – then, a nondeterministic choice between: (a) waiting a step because the channel is unready; (b) sending the message
  – if the latter, with probability 0.99 send successfully and stop
  – and with probability 0.01, message sending fails, restart



Modelling MDPs

- Guarded Commands modelling language
  - simple, textual, state-based language
  - based on Reactive Modules [AH99]
  - basic components: modules, variables and commands
- Modules:
  - components of system being modelled
  - a module represents a single MDP

```
module example
    ...
endmodule
```

Modelling MDPs

- Guarded Commands modelling language
  - simple, textual, state-based language
  - based on Reactive Modules [AH99]
  - basic components: modules, variables and commands
- Variables:
  - finite-domain (bounded integer ranges or Booleans)
  - local or global – anyone can read, only owner can modify
  - variable valuation = state of the MDP

```
module example
    s : [0..3] init 0;
    ...
endmodule
```

Modelling MDPs

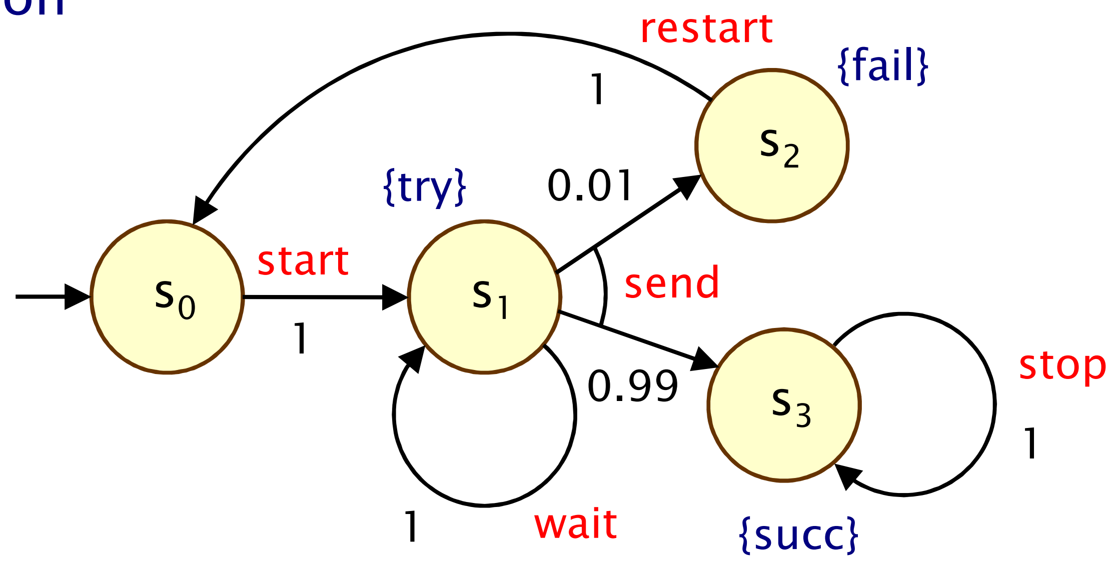
- Guarded Commands modelling language
  - simple, textual, state-based language
  - based on Reactive Modules [AH99]
  - basic components: modules, variables and commands
- Commands:
  - describe the transitions between the states

$$\underbrace{\texttt{[act]}}_{\text{action}}\ \underbrace{\texttt{exp}}_{\text{guard}}\ \texttt{->}\ \underbrace{\texttt{p}_1}_{\text{probability}}\texttt{:}\ \underbrace{\texttt{asgn}_{11}\ \&\ \texttt{asgn}_{12}\ \&\ \texttt{...}}_{\text{update}}\ \texttt{+ ... +}\ \underbrace{\texttt{p}_n}_{\text{probability}}\texttt{:}\ \underbrace{\texttt{asgn}_{n1}\ \&\ \texttt{...}}_{\text{update}}\ \texttt{;}$$

```
module example
    s : [0..3] init 0;
    ...
    [send] (s = 1) -> 0.01: (s' = 2) + 0.99: (s' = 3);
    ...
endmodule
```

Simple MDP example

- Simple communication protocol

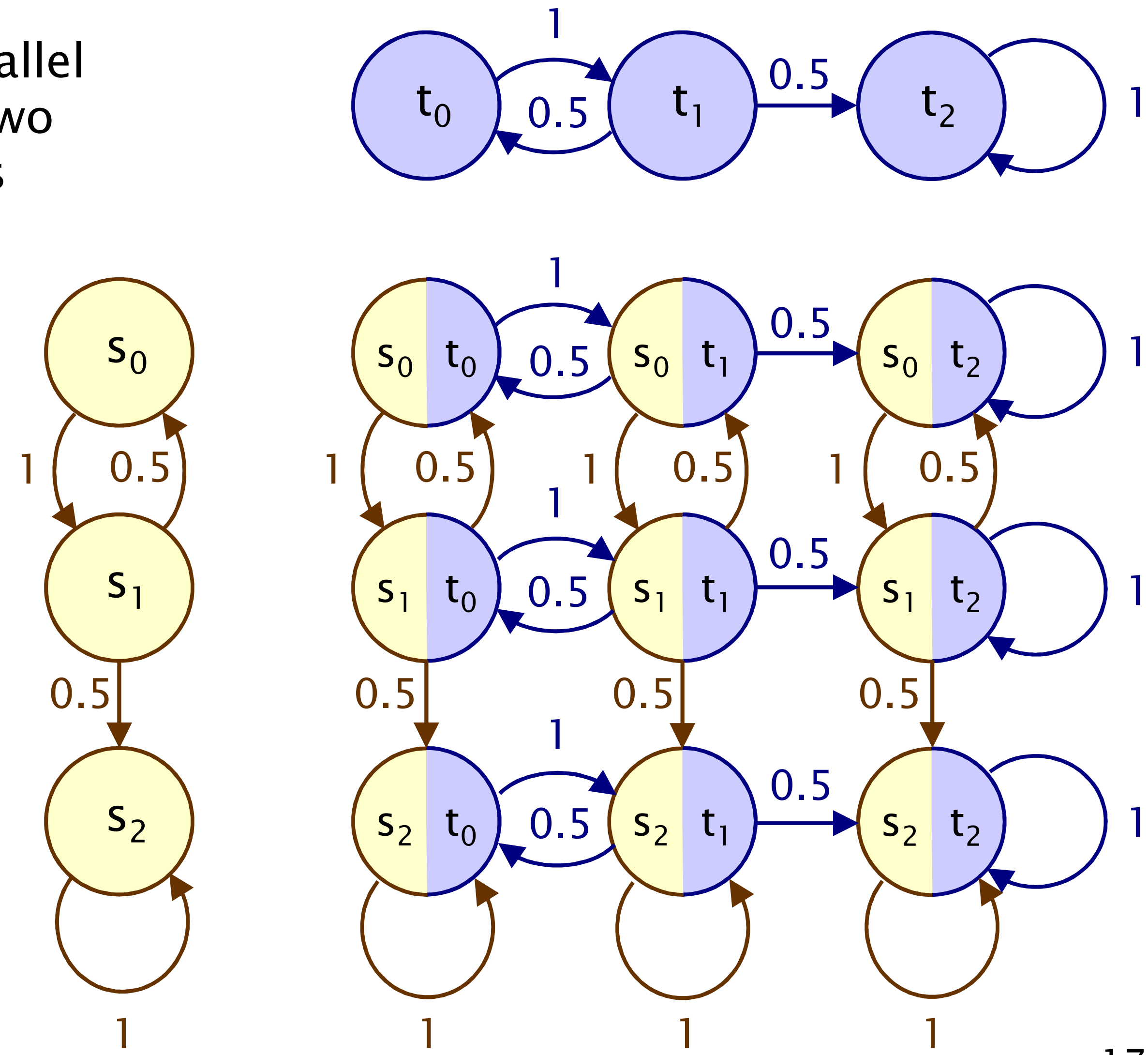



```
module example
   s : [0..3] init 0;
   [start]   (s = 0) -> (s' = 1);
   [wait]    (s = 1) -> true;
   [send]    (s = 1) -> 0.01: (s' = 2) + 0.99: (s' = 3);
   [restart] (s = 2) -> (s' = 0);
   [stop]    (s = 3) -> true;
endmodule
```

Example – Parallel composition

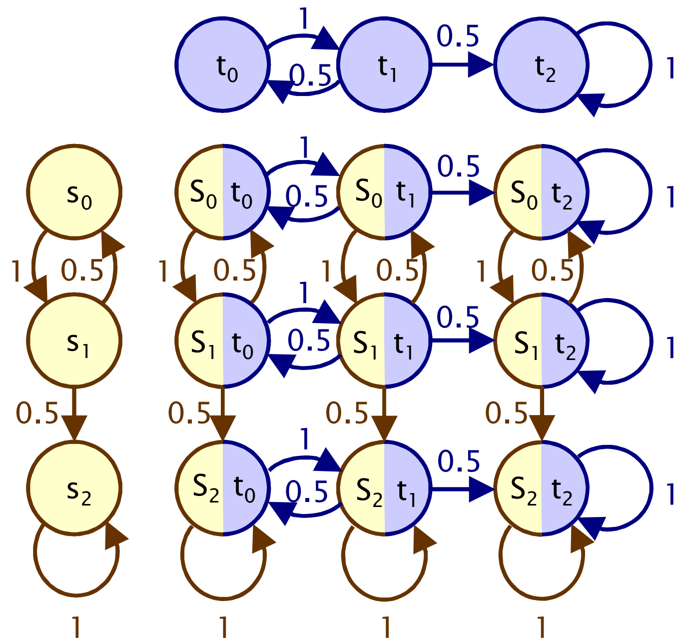
Asynchronous parallel composition of two 3-state DTMCs

Action labels omitted here



Example – Parallel composition

Asynchronous parallel composition of two 3-state DTMCs




```
module threestate
    s : [0..2] init 0;
    [] s=0 -> (s'=1);
    [] s=1 -> 0.5: (s'=s-1)
            + 0.5: (s'=s+1);
    [] s>1 -> true;
endmodule

module copy = threestate[s=t] endmodule

system
    threestate || copy
endsystem
```

Default parallel composition on matching action labels – can be omitted

Paths and probabilities

- A (finite or infinite) path through an MDP
  - is a sequence of states and action/distribution pairs
  - e.g. $s_0(a_0,\mu_0)s_1(a_1,\mu_1)s_2\ldots$
  - such that $(a_i,\mu_i) \in \mathbf{Steps}(s_i)$ and $\mu_i(s_{i+1}) > 0$ for all $i \geq 0$
  - represents an execution (i.e. one possible behaviour) of the system which the MDP is modelling
  - note that a path resolves both types of choices: nondeterministic and probabilistic

- To consider the probability of some behaviour of the MDP
  - first need to resolve the nondeterministic choices
  - …which results in a Markov chain (DTMC)
  - …for which we can define a probability measure over paths

Overview (Part 1)

- Markov decision processes (MDPs)
- Adversaries
- PCTL
- PCTL model checking
- Costs and rewards
- Case study: Firewire root contention

Adversaries

- An adversary resolves nondeterministic choice in an MDP
  - also known as “schedulers”, “strategies” or “policies”

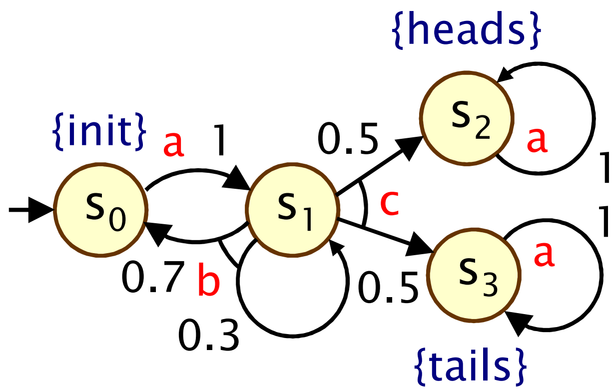
- Formally:
  - an adversary A of an MDP M is a function mapping every finite path $\omega = s_0(a_1,\mu_1)s_1 \ldots s_n$ to an element of **Steps**(s_n)

- For each A can define a probability measure Pr^A_s over paths
  - constructed through an infinite state Markov chain (DTMC)
  - states of the DTMC are the finite paths of A starting in state s
  - initial state is s (the path starting in s of length 0)
  - $\mathbf{P}^A_s(\omega,\omega') = \mu(s)$ if $\omega' = \omega(a, \mu)s$ and $A(\omega) = (a,\mu)$
  - $\mathbf{P}^A_s(\omega,\omega') = 0$ otherwise

Adversaries – Examples

- Consider the simple MDP below
  - note that s_1 is the only state for which $|\mathbf{Steps}(s)| > 1$
  - i.e. s_1 is the only state for which an adversary makes a choice
  - let μ_b and μ_c denote the probability distributions associated with actions b and c in state s_1

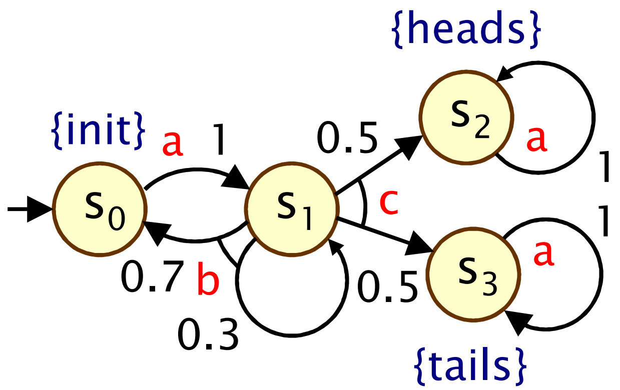
- Adversary A_1
  - picks action c the first time
  - $A_1(s_0 s_1)=(c,\mu_c)$

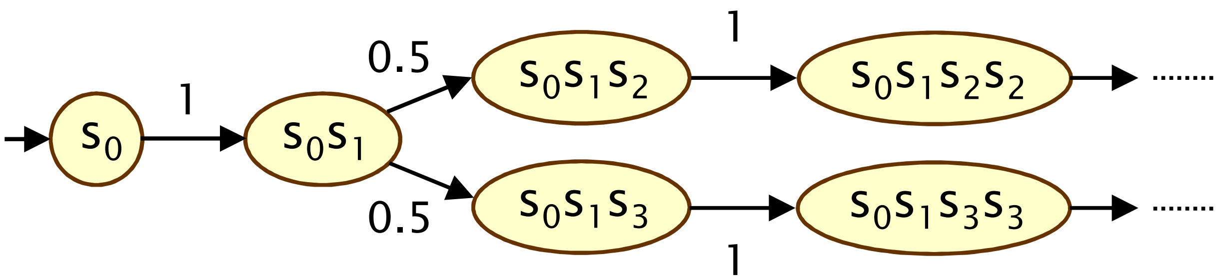


- Adversary A_2
  - picks action b the first time, then c
  - $A_2(s_0 s_1)=(b,\mu_b)$, $A_2(s_0 s_1 s_1)=(c,\mu_c)$, $A_2(s_0 s_1 s_0 s_1)=(c,\mu_c)$

Adversaries – Examples

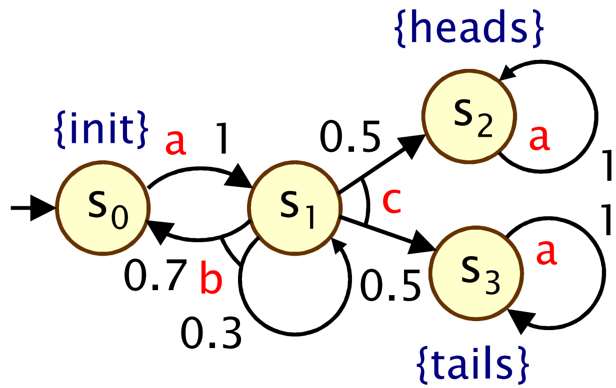
- Fragment of DTMC for adversary A_1
  - A_1 picks action c the first time

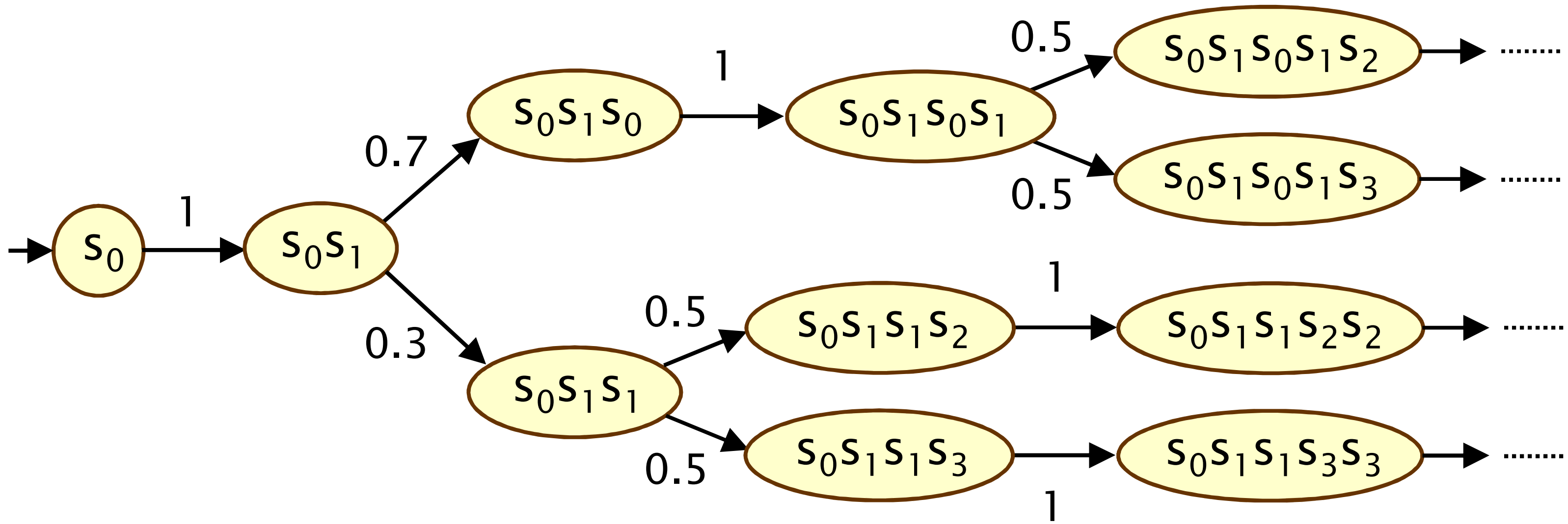




Adversaries – Examples

- Fragment of DTMC for adversary A_2
  - A_2 picks action b, then c

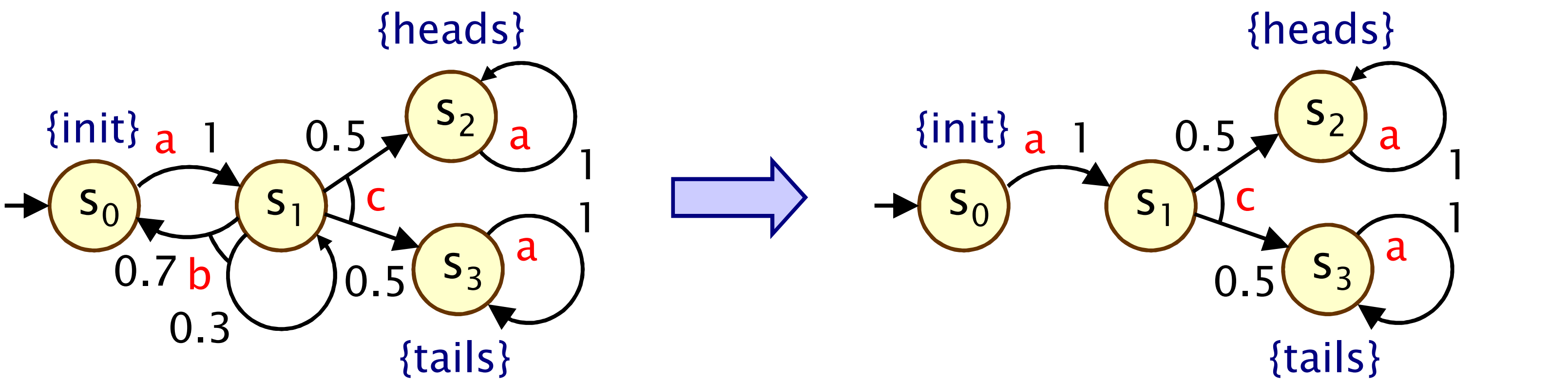




Memoryless adversaries

- Memoryless adversaries always pick same choice in a state
  - also known as: positional, Markov, simple
  - formally, for adversary A:
  - $A(s_0(a_1,\mu_1)s_1 \ldots s_n)$ depends only on s_n
  - resulting DTMC can be mapped to a |S|-state DTMC

- From previous example:
  - adversary A_1 (picks c in s_1) is memoryless, A_2 is not



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PCTL

- Temporal logic for describing properties of MDPs
  - PCTL = Probabilistic Computation Tree Logic [HJ94]
  - essentially the same as the logic pCTL of [ASB+95]
- Extension of (non-probabilistic) temporal logic CTL
  - key addition is probabilistic operator P
  - quantitative extension of CTL's A and E operators
- Example
  - send → $P_{\geq 0.95}$ [true $U^{\leq 10}$ deliver]
  - "if a message is sent, then the probability of it being delivered within 10 steps is at least 0.95"

PCTL syntax

ψ is true with probability ~p

- PCTL syntax:
  - $\phi ::= \text{true} \mid a \mid \phi \wedge \phi \mid \neg\phi \mid P_{\sim p} [\, \psi \,]$ (state formulas)
  - $\psi ::= X\, \phi \mid \phi\, U^{\leq k}\, \phi \mid \phi\, U\, \phi$ (path formulas)

"next" "bounded until" "until"

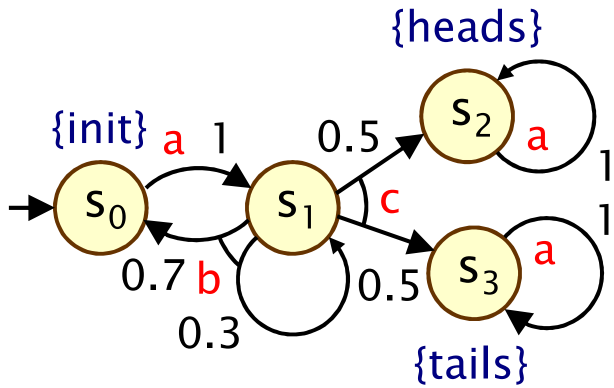
  - where a is an atomic proposition, used to identify states of interest, $p \in [0,1]$ is a probability, $\sim \in \{<,>,\leq,\geq\}$, $k \in \mathbb{N}$
- A PCTL formula is always a state formula
  - path formulas only occur inside the P operator

PCTL semantics for MDPs

- PCTL formulas interpreted over states of an MDP
  - $s \models \phi$ denotes ϕ is "true in state s" or "satisfied in state s"

- Semantics of (non-probabilistic) state formulas:
  - for a state s of the MDP (S,s_{init},P,L):
  - $s \models a \quad \Leftrightarrow \quad a \in L(s)$
  - $s \models \phi_1 \wedge \phi_2 \quad \Leftrightarrow \quad s \models \phi_1$ and $s \models \phi_2$
  - $s \models \neg\phi \quad \Leftrightarrow \quad s \models \phi$ is false

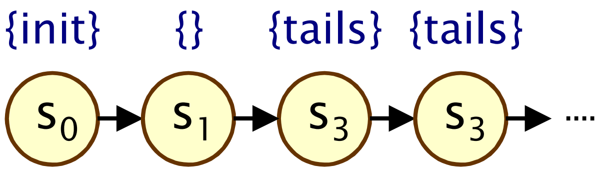
- Examples
  - $s_3 \models$ tails
  - $s_2 \models$ heads $\wedge \neg$init



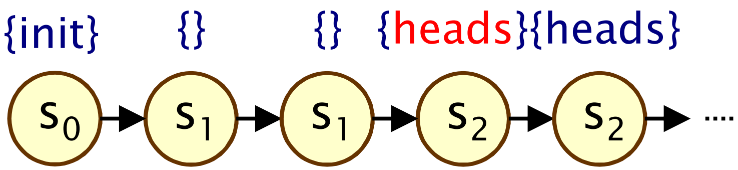
PCTL semantics for MDPs

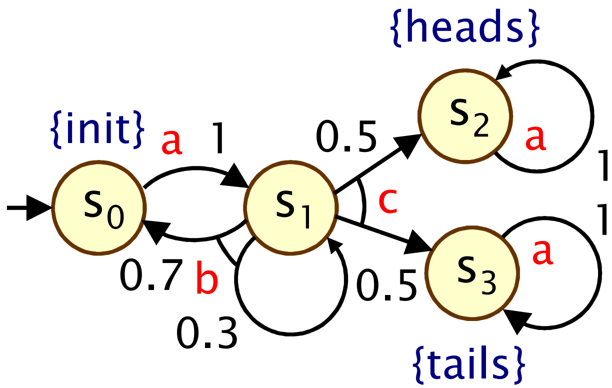
- Semantics of path formulas:
  - for a path $\omega = s_0 s_1 s_2 \ldots$ in the MDP:
  - $\omega \vDash X\ \phi \quad \Leftrightarrow \quad s_1 \vDash \phi$
  - $\omega \vDash \phi_1\ U^{\leq k}\ \phi_2 \quad \Leftrightarrow \quad \exists i \leq k$ such that $s_i \vDash \phi_2$ and $\forall j<i,\ s_j \vDash \phi_1$
  - $\omega \vDash \phi_1\ U\ \phi_2 \quad \Leftrightarrow \quad \exists k \geq 0$ such that $\omega \vDash \phi_1\ U^{\leq k}\ \phi_2$

- Some examples of satisfying paths:
  - $X\ \neg init$



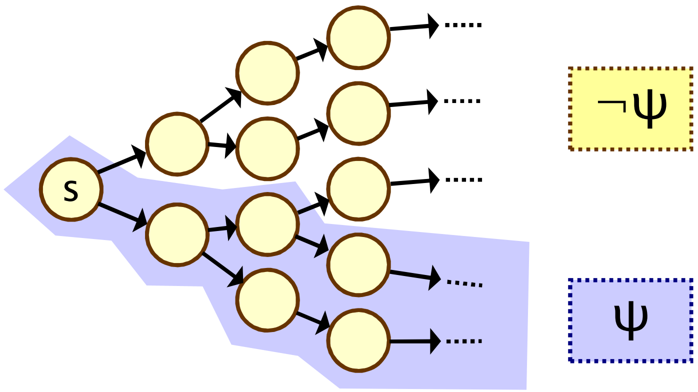
  - $\neg tails\ U\ heads$





PCTL semantics for MDPs

- Semantics of the probabilistic operator P
  - can only define probabilities for a specific adversary A
  - s ⊨ $P_{\sim p}$ [ψ] means "the probability, from state s, that ψ is true for an outgoing path satisfies ~p for all adversaries A"
  - formally s ⊨ $P_{\sim p}$ [ψ] ⇔ $Prob^A(s, \psi) \sim p$ for all adversaries A
  - where $Prob^A(s, \psi) = Pr^A_s \{ \omega \in Path^A(s) \mid \omega \models \psi \}$



$Prob^A(s, \psi) \sim p$

Minimum and maximum probabilities

- Letting:
  - $p_{max}(s, \psi) = \sup_A Prob^A(s, \psi)$
  - $p_{min}(s, \psi) = \inf_A Prob^A(s, \psi)$
- We have:
  - if $\sim \in \{\geq,>\}$, then $s \models P_{\sim p} [\ \psi\ ] \quad \Leftrightarrow \quad p_{min}(s, \psi) \sim p$
  - if $\sim \in \{<,\leq\}$, then $s \models P_{\sim p} [\ \psi\ ] \quad \Leftrightarrow \quad p_{max}(s, \psi) \sim p$
- Model checking $P_{\sim p}[\ \psi\ ]$ reduces to the computation over all adversaries of either:
  - the minimum probability of ψ holding
  - the maximum probability of ψ holding
- Crucial result for model checking PCTL on MDPs
  - memoryless adversaries suffice, i.e. there are always memoryless adversaries A_{min} and A_{max} for which:
  - $Prob^{Amin}(s, \psi) = p_{min}(s, \psi)$ and $Prob^{Amax}(s, \psi) = p_{max}(s, \psi)$

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- Markov decision processes (MDPs)
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PCTL model checking

- Algorithm for PCTL model checking [BdA95]
  - inputs: MDP $M=(S,s_{init},Steps,L)$, PCTL formula ϕ
  - output: $Sat(\phi) = \{ s \in S \mid s \models \phi \}$ = set of states satisfying ϕ

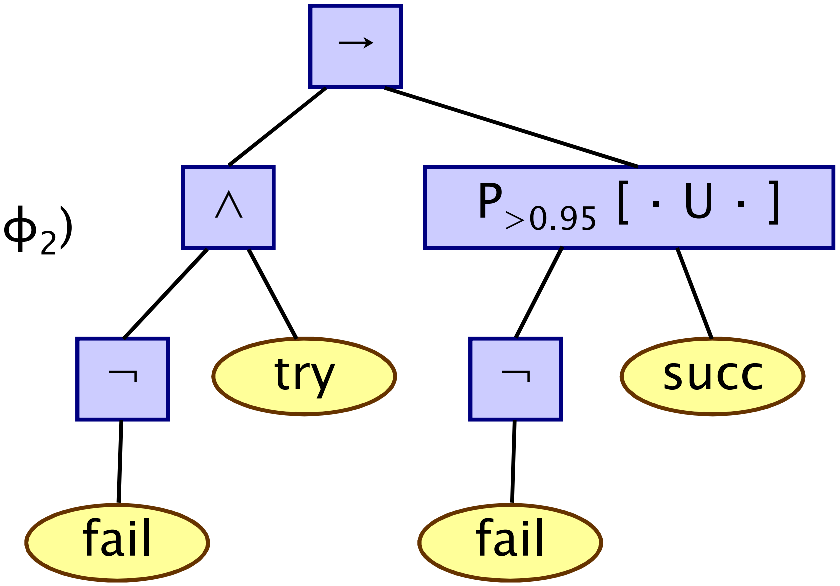
- What does it mean for an MDP D to satisfy a formula ϕ?
  - sometimes, want to check that $s \models \phi \ \forall\ s \in S$, i.e. $Sat(\phi) = S$
  - sometimes, just want to know if $s_{init} \models \phi$, i.e. if $s_{init} \in Sat(\phi)$

- Sometimes, focus on quantitative results
  - e.g. compute result of Pmax=? [F error]
  - e.g. compute result of Pmax=? [$F^{\leq k}$ error] for $0 \leq k \leq 100$

PCTL model checking for MDPs

- Basic algorithm proceeds by induction on parse tree of ϕ
  - example: $\phi = (\neg\text{fail} \wedge \text{try}) \rightarrow P_{>0.95}\ [\ \neg\text{fail}\ U\ \text{succ}\ ]$

- For the non-probabilistic operators:
  - $\text{Sat}(\text{true}) = S$
  - $\text{Sat}(a) = \{\ s \in S \mid a \in L(s)\ \}$
  - $\text{Sat}(\neg\phi) = S \setminus \text{Sat}(\phi)$
  - $\text{Sat}(\phi_1 \wedge \phi_2) = \text{Sat}(\phi_1) \cap \text{Sat}(\phi_2)$

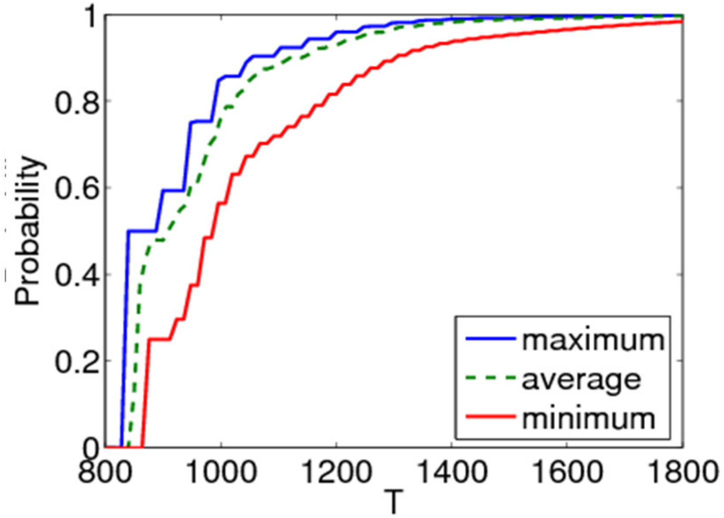


- For the $P_{\sim p}\ [\ \psi\ ]$ operator
  - need to compute the probabilities $\text{Prob}(s, \psi)$ for all states $s \in S$
  - focus here on "until" case: $\psi = \phi_1\ U\ \phi_2$

Quantitative properties

- For PCTL properties with P as the outermost operator
  - quantitative form (two types): $\mathrm{Pmin}_{=?}$ [ψ] and $\mathrm{Pmax}_{=?}$ [ψ]
  - i.e. "what is the minimum/maximum probability (over all adversaries) that path formula ψ is true?"
  - corresponds to an analysis of best-case or worst-case behaviour of the system
  - model checking is no harder since compute the values of $p_{min}(s, \psi)$ or $p_{max}(s, \psi)$ anyway
  - useful to spot patterns/trends

- Example: CSMA/CD protocol
  - "min/max probability that a message is sent within the deadline"



Some real PCTL examples

- Byzantine agreement protocol
  - $\mathrm{Pmin}_{=?}$ [F (agreement $\wedge$ rounds$\leq$2)]
  - "what is the minimum probability that agreement is reached within two rounds?"
- CSMA/CD communication protocol
  - $\mathrm{Pmax}_{=?}$ [F collisions=k]
  - "what is the maximum probability of k collisions?"
- Self-stabilisation protocols
  - $\mathrm{Pmin}_{=?}$ [$F^{\leq t}$ stable]
  - "what is the minimum probability of reaching a stable state within k steps?"

PCTL until for MDPs

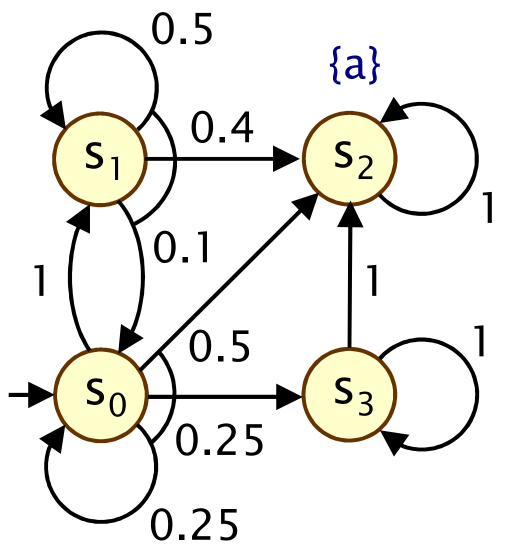
- Computation of probabilities $p_{min}(s, \phi_1 \text{ U } \phi_2)$ for all $s \in S$
- First identify all states where the probability is 1 or 0
  - "precomputation" algorithms, yielding sets S^{yes}, S^{no}
- Then compute (min) probabilities for remaining states ($S^?$)
  - either: solve linear programming problem
  - or: approximate with an iterative solution method

Example:

$P_{\geq p}$ [F a]

$\equiv$

$P_{\geq p}$ [true U a]

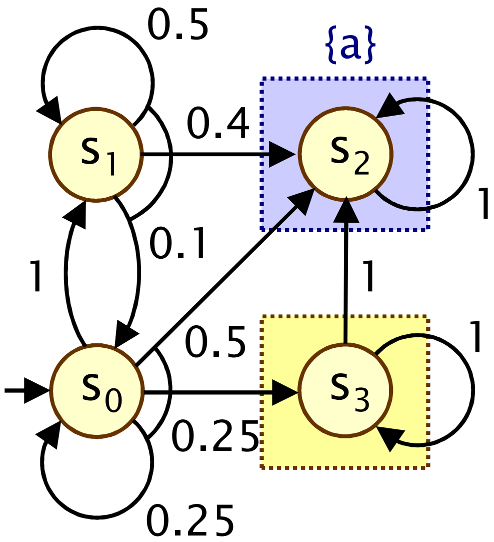


PCTL until – Precomputation

- Identify all states where $p_{min}(s, \phi_1 \text{ U } \phi_2)$ is 1 or 0
  - S^{yes} = $\text{Sat}(P_{\geq 1} [\ \phi_1 \text{ U } \phi_2\ ])$, S^{no} = $\text{Sat}(\neg\ P_{>0} [\ \phi_1 \text{ U } \phi_2\ ])$
- Two graph-based precomputation algorithms:
  - algorithm Prob1A computes S^{yes}
    - for all adversaries the probability of satisfying ϕ_1 U ϕ_2 is 1
  - algorithm Prob0E computes S^{no}
    - there exists an adversary for which the probability is 0

Example:

$P_{\geq p}$ [F a]



$S^{yes} = \text{Sat}(P_{\geq 1}\ [\ F\ a\ ])$

$S^{no} = \text{Sat}(\neg P_{>0}\ [\ F\ a\ ])$

Method 1 – Linear programming

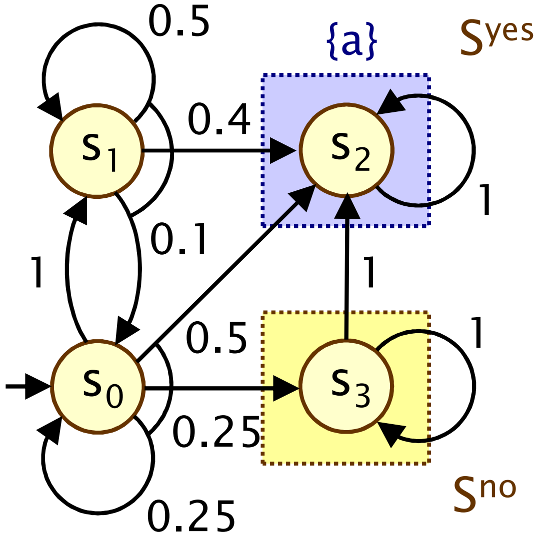
- Probabilities $p_{min}(s, \phi_1 \text{ U } \phi_2)$ for remaining states in the set $S^? = S \setminus (S^{yes} \cup S^{no})$ can be obtained as the unique solution of the following linear programming (LP) problem:

$$\text{maximize } \sum_{s \in S^?} x_s \text{ subject to the constraints:}$$
$$x_s \leq \sum_{s' \in S^?} \mu(s') \cdot x_{s'} + \sum_{s' \in S^{yes}} \mu(s')$$
$$\text{for all } s \in S^? \text{ and for all } (a, \mu) \in \mathbf{Steps}(s)$$

- Simple case of a more general problem known as the stochastic shortest path problem [BT91]

- This can be solved with standard techniques
  - e.g. Simplex, ellipsoid method, branch-and-cut

Example – PCTL until (LP)



Let $x_i = p_{min}(s_i, F\ a)$

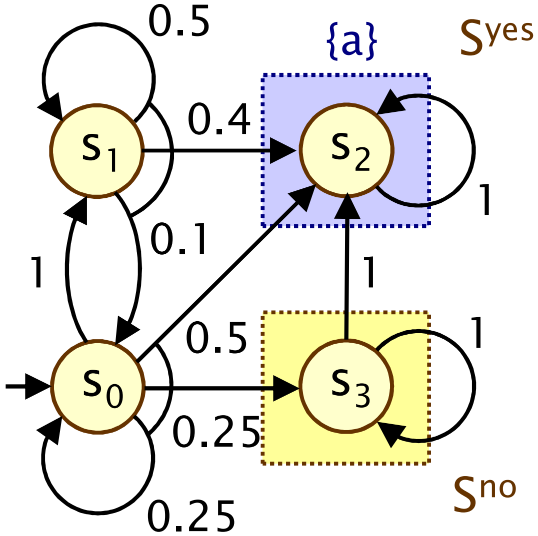
S^{yes}: $x_2=1$, S^{no}: $x_3=0$

For $S^? = \{x_0, x_1\}$:

Maximise x_0+x_1 subject to constraints:

- $x_0 \leq x_1$
- $x_0 \leq 0.25 \cdot x_0 + 0.5$
- $x_1 \leq 0.1 \cdot x_0 + 0.5 \cdot x_1 + 0.4$

Example – PCTL until (LP)



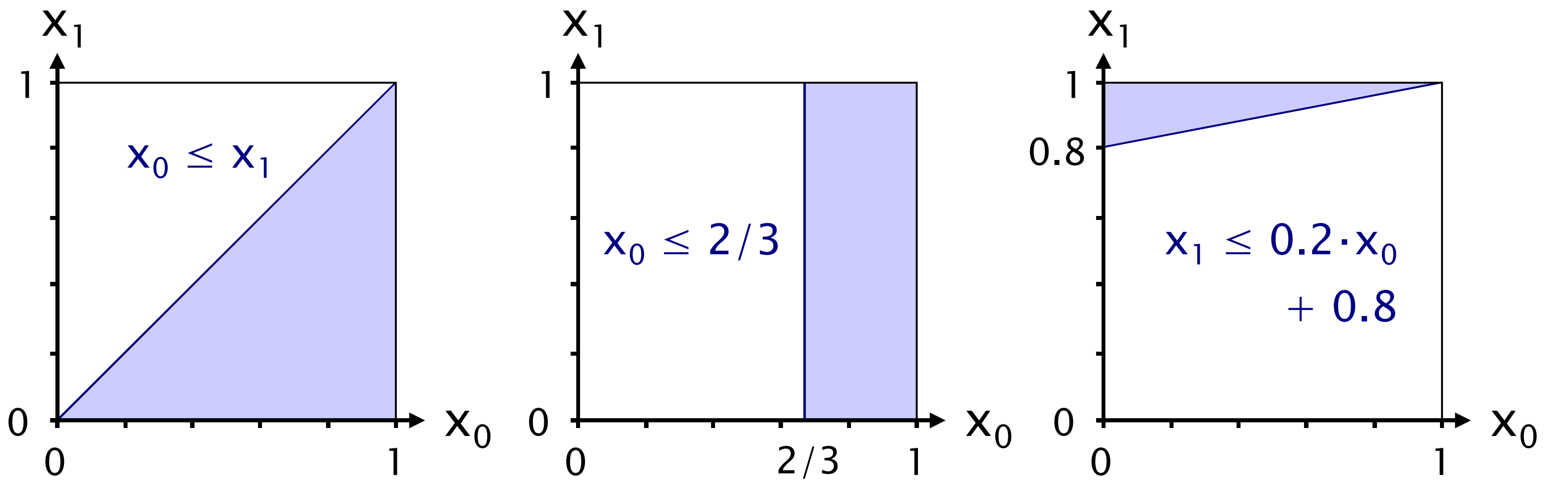
Let $x_i = p_{min}(s_i, \text{F } a)$

S^{yes}: $x_2=1$, S^{no}: $x_3=0$

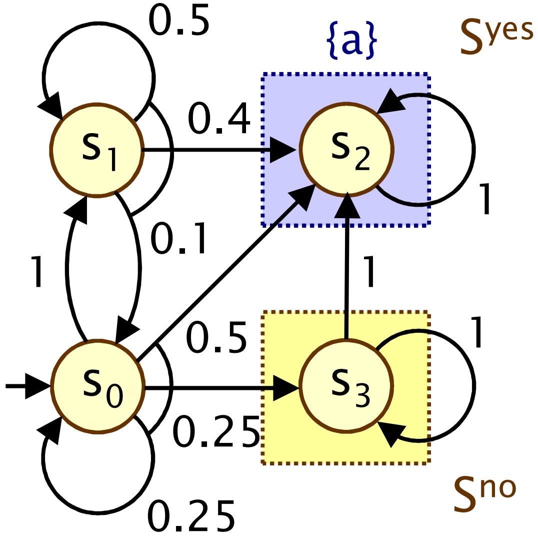
For $S^? = \{x_0, x_1\}$:

Maximise x_0+x_1 subject to constraints:

- $x_0 \leq x_1$
- $x_0 \leq 2/3$
- $x_1 \leq 0.2 \cdot x_0 + 0.8$



Example – PCTL until (LP)



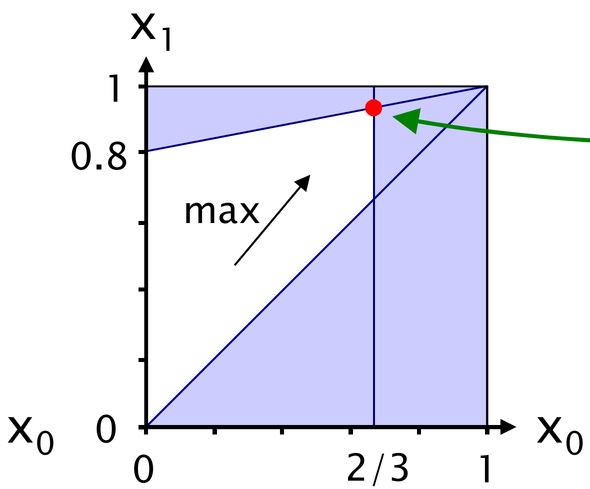
Let $x_i = p_{min}(s_i, \text{F } a)$

S^{yes}: $x_2=1$, S^{no}: $x_3=0$

For $S^? = \{x_0, x_1\}$:

Maximise x_0+x_1 subject to constraints:

- $x_0 \leq x_1$
- $x_0 \leq 2/3$
- $x_1 \leq 0.2 \cdot x_0 + 0.8$



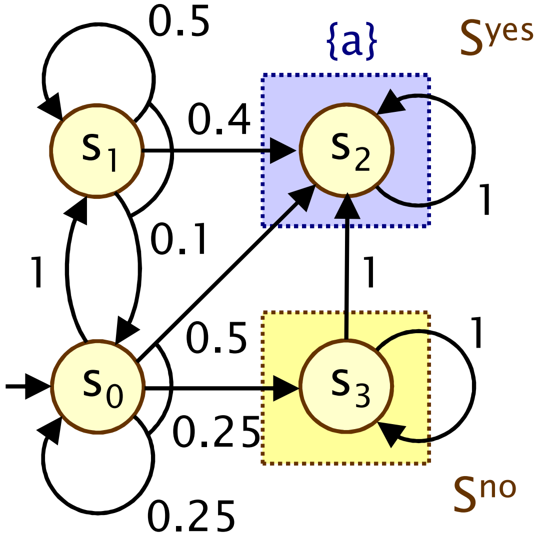
Solution:

(x_0, x_1)

$=$

$(2/3, 14/15)$

Example – PCTL until (LP)



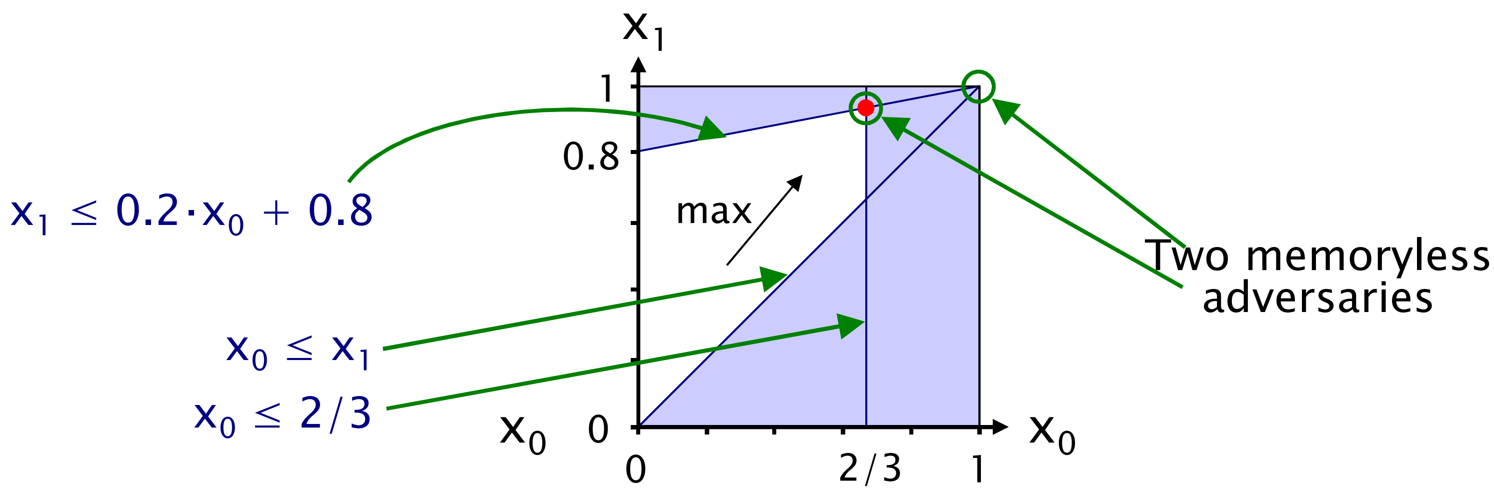
Let $x_i = p_{min}(s_i, F\ a)$

S^{yes}: $x_2=1$, S^{no}: $x_3=0$

For $S^? = \{x_0, x_1\}$:

Maximise x_0+x_1 subject to constraints:

- $x_0 \leq x_1$
- $x_0 \leq 2/3$
- $x_1 \leq 0.2 \cdot x_0 + 0.8$



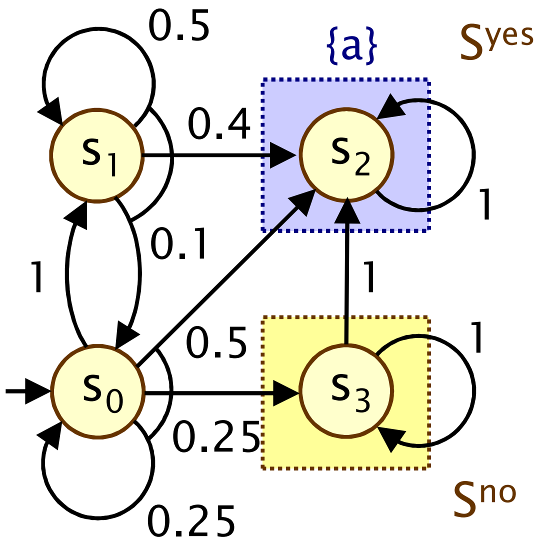
Method 2 – Value iteration

- For probabilities $p_{min}(s, \phi_1\ U\ \phi_2)$ it can be shown that:
  - $p_{min}(s, \phi_1\ U\ \phi_2) = \lim_{n\to\infty} x_s^{(n)}$ where:

$$x_s^{(n)} = \begin{cases} 1 & \text{if } s \in S^{yes} \\ 0 & \text{if } s \in S^{no} \\ 0 & \text{if } s \in S^{?} \text{ and } n = 0 \\ \min_{(a,\mu)\in Steps(s)} \left(\sum_{s'\in S} \mu(s') \cdot x_{s'}^{(n-1)} \right) & \text{if } s \in S^{?} \text{ and } n > 0 \end{cases}$$

- This forms the basis for an (approximate) iterative solution
  - iterations terminated when solution converges sufficiently

Example – PCTL until (value iteration)



Compute: $p_{min}(s_i, F\ a)$

$S^{yes} = \{x_2\}$, $S^{no} = \{x_3\}$, $S^{?} = \{x_0, x_1\}$

$[\ x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\ ]$

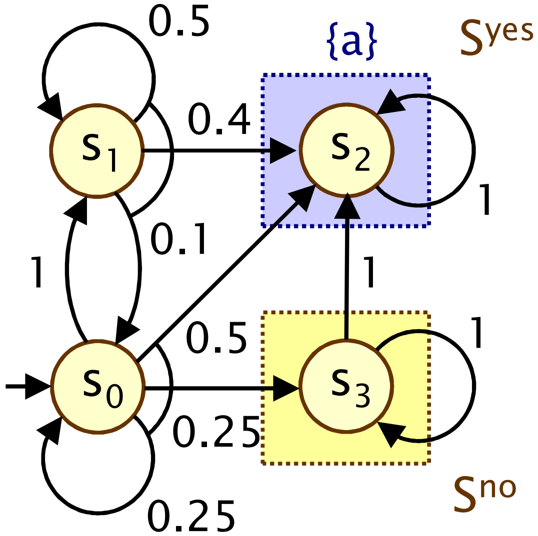
n=0: $[\ 0, 0, 1, 0\ ]$

n=1: $[\ \min(0, 0.25\cdot 0+0.5),$
$0.1\cdot 0+0.5\cdot 0+0.4, 1, 0\ ]$
$= [\ 0, 0.4, 1, 0\ ]$

n=2: $[\ \min(0.4, 0.25\cdot 0+0.5),$
$0.1\cdot 0+0.5\cdot 0.4+0.4, 1, 0\ ]$
$= [\ 0.4, 0.6, 1, 0\ ]$

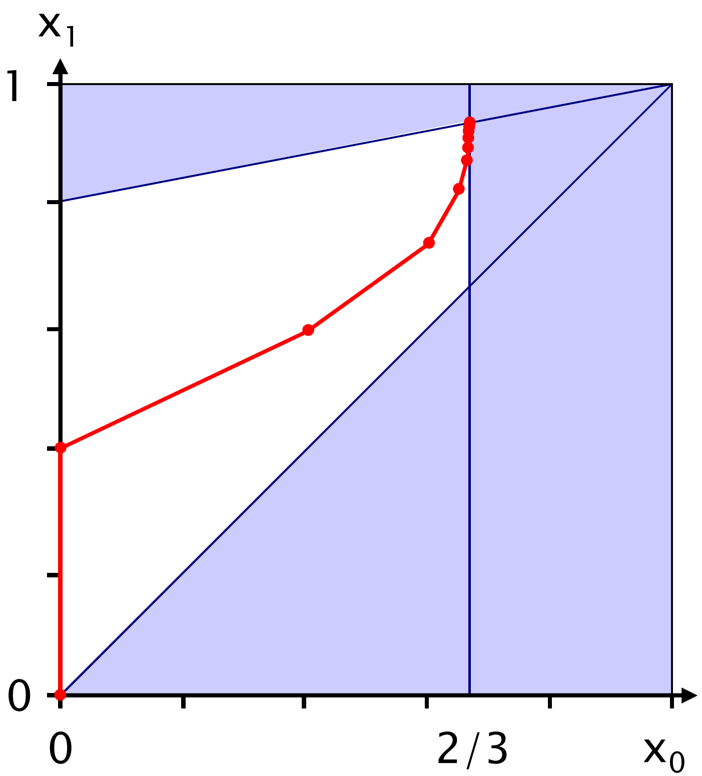
n=3: …

Example – PCTL until (value iteration)



	$[\ x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\ ]$
n=0:	[0.000000, 0.000000, 1, 0]
n=1:	[0.000000, 0.400000, 1, 0]
n=2:	[0.400000, 0.600000, 1, 0]
n=3:	[0.600000, 0.740000, 1, 0]
n=4:	[0.650000, 0.830000, 1, 0]
n=5:	[0.662500, 0.880000, 1, 0]
n=6:	[0.665625, 0.906250, 1, 0]
n=7:	[0.666406, 0.919688, 1, 0]
n=8:	[0.666602, 0.926484, 1, 0]
n=9:	[0.666650, 0.929902, 1, 0]
	…
n=20:	[0.666667, 0.933332, 1, 0]
n=21:	[0.666667, 0.933332, 1, 0]
	$\approx$ [2/3, 14/15, 1, 0]

Example – Value iteration + LP



	$[\ x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\ ]$
n=0:	[0.000000, 0.000000, 1, 0]
n=1:	[0.000000, 0.400000, 1, 0]
n=2:	[0.400000, 0.600000, 1, 0]
n=3:	[0.600000, 0.740000, 1, 0]
n=4:	[0.650000, 0.830000, 1, 0]
n=5:	[0.662500, 0.880000, 1, 0]
n=6:	[0.665625, 0.906250, 1, 0]
n=7:	[0.666406, 0.919688, 1, 0]
n=8:	[0.666602, 0.926484, 1, 0]
n=9:	[0.666650, 0.929902, 1, 0]
	…
n=20:	[0.666667, 0.933332, 1, 0]
n=21:	[0.666667, 0.933332, 1, 0]

$\approx$ [2/3, 14/15, 1, 0]

PCTL model checking - Summary

- Computation of set Sat(Φ) for MDP M and PCTL formula Φ
  - recursive descent of parse tree
  - combination of graph algorithms, numerical computation
- Probabilistic operator P:
  - X Φ : one matrix-vector multiplication, $O(|S|^2)$
  - $\Phi_1 \ U^{\leq k} \ \Phi_2$: k matrix-vector multiplications, $O(k|S|^2)$
  - $\Phi_1 \ U \ \Phi_2$: linear programming problem, polynomial in |S| (assuming use of linear programming)
- Complexity:
  - linear in |Φ| and polynomial in |S|
  - S is states in MDP, assume |Steps(s)| is constant

Overview (Part 1)

- Markov decision processes (MDPs)
- Adversaries
- PCTL
- PCTL model checking
- Costs and rewards
- Case study: Firewire root contention

Costs and rewards

- We augment DTMCs with rewards (or, conversely, costs)
  - real-valued quantities assigned to states and/or transitions
  - these can have a wide range of possible interpretations

- Some examples:
  - elapsed time, power consumption, size of message queue, number of messages successfully delivered, net profit, …

- Costs? or rewards?
  - mathematically, no distinction between rewards and costs
  - when interpreted, we assume that it is desirable to minimise costs and to maximise rewards
  - we will consistently use the terminology “rewards” regardless

Reward-based properties

- Properties of MDPs augmented with rewards
  - allow a wide range of quantitative measures of the system
  - basic notion: expected value of rewards
  - formal property specifications will be in an extension of PCTL

- More precisely, we use two distinct classes of property…

- Instantaneous properties
  - the expected value of the reward at some time point

- Cumulative properties
  - the expected cumulated reward over some period

PCTL and rewards

- Extend PCTL to incorporate reward-based properties
  - add an R operator, which is similar to the existing P operator

expected reward is ~r

  - ϕ ::= … | $P_{\sim p}$ [ψ] | $R_{\sim r}$ [$I^{=k}$] | $R_{\sim r}$ [$C^{\leq k}$] | $R_{\sim r}$ [F ϕ]

"instantaneous" "cumulative" "reachability"

  - where $r \in \mathbb{R}_{\geq 0}$, $\sim \in \{<,>,\leq,\geq\}$, $k \in \mathbb{N}$

- $R_{\sim r}$ [·] means "the expected value of · satisfies ~r"

Types of reward formulas

- Instantaneous: $R_{\sim r}$ [$I^{=k}$]
  - "the expected value of the state reward at time-step k is ~r"
  - e.g. "the expected queue size after exactly 90 seconds"

- Cumulative: $R_{\sim r}$ [$C^{\leq k}$]
  - "the expected reward cumulated up to time-step k is ~r"
  - e.g. "the expected power consumption over one hour"

- Reachability: $R_{\sim r}$ [F ϕ]
  - "the expected reward cumulated before reaching a state satisfying ϕ is ~r"
  - e.g. "the expected time for the algorithm to terminate"

Model checking MDP reward formulas

- Instantaneous: $R_{\sim r}$ [$I^{=k}$]
  - similar to the computation of bounded until probabilities
  - solution of recursive equations
- Cumulative: $R_{\sim r}$ [$C^{\leq k}$]
  - extension of bounded until computation
  - solution of recursive equations
- Reachability: $R_{\sim r}$ [F ϕ]
  - similar to the case for P operator and until
  - graph-based precomputation (identify ∞-reward states)
  - then linear programming problem (or value iteration)

Summary

- Markov decision processes (MDPs)
  - probabilistic as well as nondeterminisitic behaviours
  - to model concurrency, underspecification, …
  - easy to model using guarded commands
- Adversaries resolve nondeterminism in an MDP
  - induce a probability space over paths
  - consider minimum/maximum probabilities over all adversaries
- Property specifications
  - probabilistic extensions of temporal logic, e.g. PCTL
  - also: expected value of costs/rewards
  - quantify over all adversaries
- Model checking algorithms
  - covered two basic techniques for MDPs: linear programming or value iteration