Lecture 16
Automata-based properties

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Property specifications

- 1. Reachability properties, e.g. in PCTL
  - F a or $F^{\leq t}$ a (reachability)
  - a U b or a $U^{\leq t}$ b (until – constrained reachability)
  - G a (invariance) (dual of reachability)
  - probability computation: graph analysis + solution of linear equation system (or linear optimisation problem)

- 2. Long-run properties, e.g. in LTL
  - GF a (repeated reachability)
  - FG a (persistence)
  - probability computation: BSCCs + probabilistic reachability

- This lecture: more expressive class for type 1

Overview

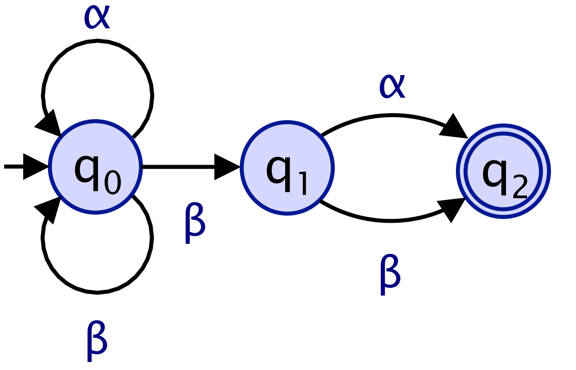
- Nondeterministic finite automata (NFA)
- Regular expressions and regular languages
- Deterministic finite automata (DFA)
- Regular safety properties
- DFAs and DTMCs

Some notation

- Let Σ be a finite alphabet
- A (finite or infinite) word w over Σ is
  - a sequence of $\alpha_1\alpha_2\ldots$ where $\alpha_i \in \Sigma$ for all i
- A prefix w' of word $w = \alpha_1\alpha_2\ldots$ is
  - a finite word $\beta_1\, \beta_2\ldots\, \beta_n$ with $\beta_i = \alpha_i$ for all $1 \leq i \leq n$
- Σ^* denotes the set of finite words over Σ
- Σ^ω denotes the set of infinite words over Σ

Finite automata

- A nondeterministic finite automaton (NFA) is…
  - a tuple $A = (Q, \Sigma, \delta, Q_0, F)$ where:
  - Q is a finite set of states
  - Σ is an alphabet
  - $\delta : Q \times \Sigma \rightarrow 2^Q$ is a transition function
  - $Q_0 \subseteq Q$ is a set of initial states
  - $F \subseteq Q$ is a set of “accept” states



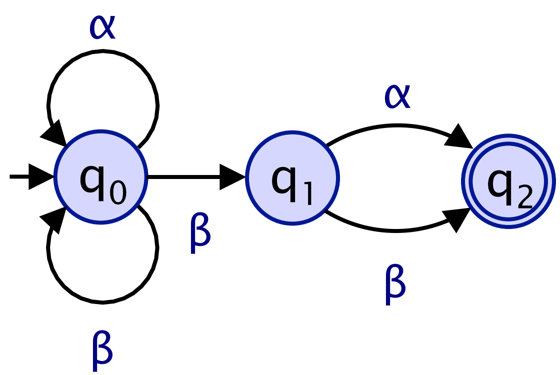
Language of an NFA

- Consider an NFA $A = (Q, \Sigma, \delta, Q_0, F)$

- A run of A on a finite word $w=\alpha_1\alpha_2\ldots\alpha_n$ is:
  - a sequence of automata states $q_0q_1\ldots q_n$ such that:
  - $q_0 \in Q_0$ and $q_{i+1} \in \delta(q_i, \alpha_{i+1})$ for all $0 \leq i < n$
- An accepting run is a run with $q_n \in F$

- Word w is accepted by A iff:
  - there exists an accepting run of A on w
- The language of A, denoted L(A) is:
  - the set of all words accepted by A
- Automata A and A' are equivalent if L(A)=L(A')

Example – NFA



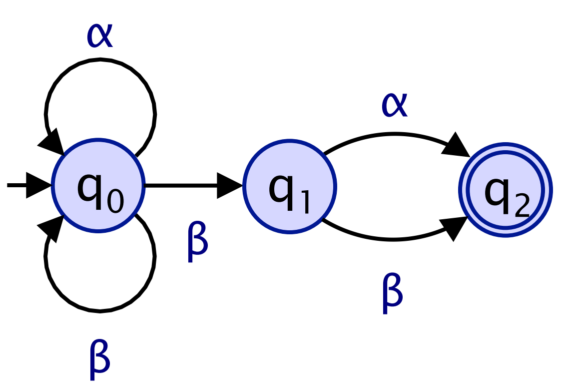
Regular expressions

- Regular expressions E over a finite alphabet Σ
  - are given by the following grammar:
  - $E ::= \emptyset \mid \varepsilon \mid \alpha \mid E + E \mid E.E \mid E^*$
  - where $\alpha \in \Sigma$

- Language $L(E) \subseteq \Sigma^*$ of a regular expression:
  - $L(\emptyset) = \emptyset$ (empty language)
  - $L(\varepsilon) = \{ \varepsilon \}$ (empty word)
  - $L(\alpha) = \{ \alpha \}$ (symbol)
  - $L(E_1 + E_2) = L(E_1) \cup L(E_2)$ (union)
  - $L(E_1.E_2) = \{ w_1.w_2 \mid w_1 \in L(E_1) \text{ and } w_2 \in L(E_2) \}$ (concatenation)
  - $L(E^*) = \{ w^i \mid w \in L(E) \text{ and } i \in \mathbb{N} \}$ (finite repetition)

Regular languages

- A set of finite words L is a regular language…
  - iff L = L(E) for some regular expression E
  - iff L = L(A) for some finite automaton A

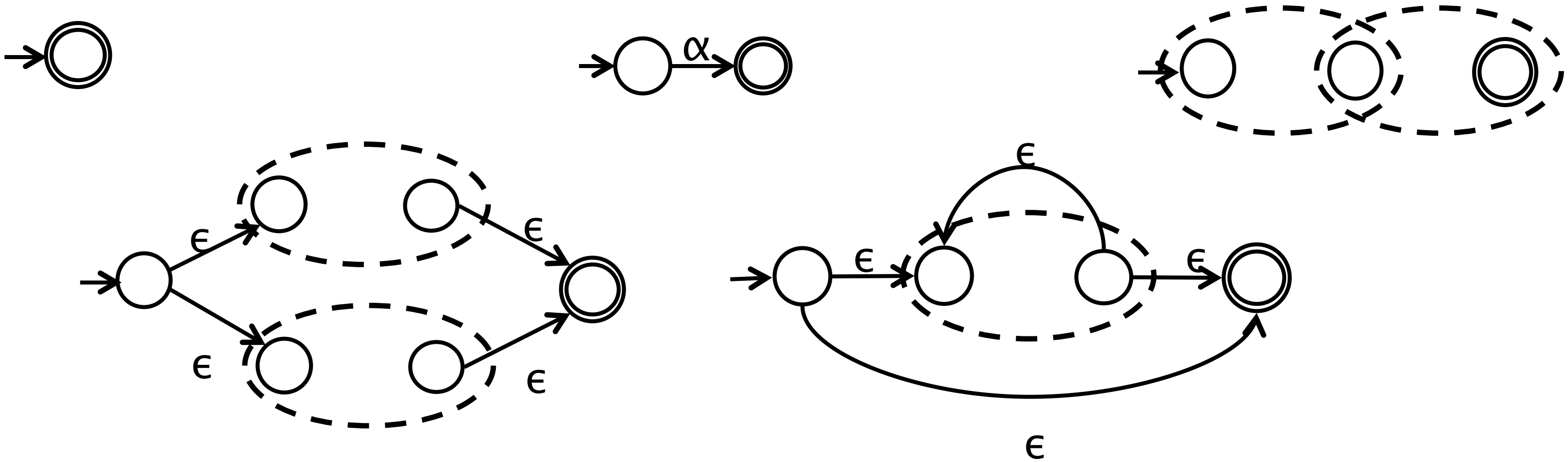


$(\alpha+\beta)^*\beta(\alpha+\beta)$

(i.e. penultimate symbol is β)

Operations on NFA

- Can construct NFA from regular expression inductively
  - includes addition (and then removal) of ε–transitions



- Can construct the intersection of two NFA
  - build (synchronised) product automaton
  - cross product of $A_1 \otimes A_2$ accepts $L(A_1) \cap L(A_2)$

Deterministic finite automata

- A finite automaton is deterministic if:
  - $|Q_0|=1$
  - $|\delta(q, \alpha)| \leq 1$ for all $q \in Q$ and $\alpha \in \Sigma$
  - i.e. one initial state and no nondeterministic successors

- A deterministic finite automaton (DFA) is total if:
  - $|\delta(q, \alpha)| = 1$ for all $q \in Q$ and $\alpha \in \Sigma$
  - i.e. unique successor states

- A total DFA
  - can always be constructed from a DFA
  - has a unique run for any word $w \in \Sigma^*$

Determinisation: NFA → DFA

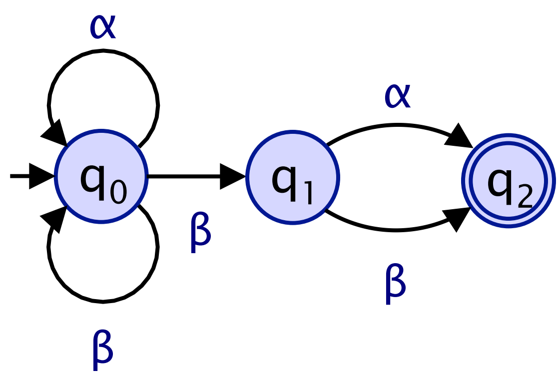
- Determinisation of an NFA $A = (Q, \Sigma, \delta, Q_0, F)$
  - i.e. removal of choice in each automata state

- Equivalent DFA is $A_{det} = (2^Q, \Sigma, \delta_{det}, q_0, F_{det})$ where:
  - $\delta_{det}(Q', \alpha) = \bigcup_{q \in Q'} \delta(q, \alpha)$
  - $F_{det} = \{ Q' \subseteq Q \mid Q' \cap F \neq \varnothing \}$

- Note exponential blow-up in size…

Example

NFA A

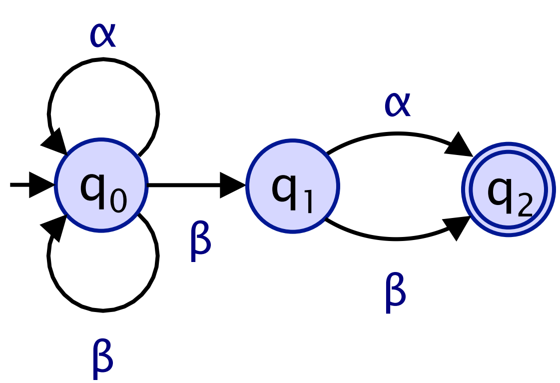


regexp:

$(\alpha+\beta)^*\beta(\alpha+\beta)$

Example

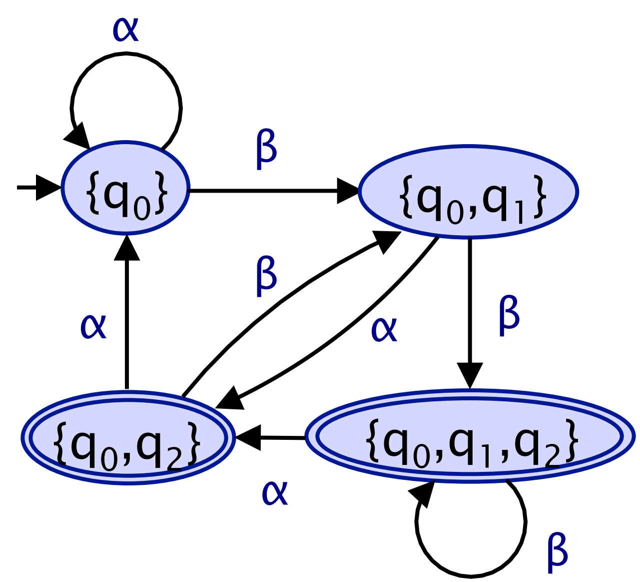
NFA A



regexp:

$(\alpha+\beta)^*\beta(\alpha+\beta)$

DFA A_{det}



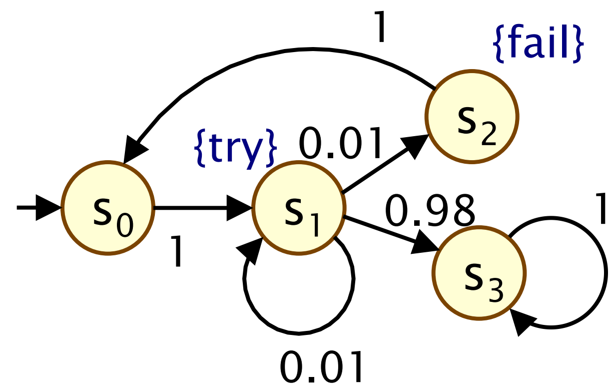
Other properties of NFA/DFA

- NFA/DFA have the same expressive power
  - but NFA can be more efficient (up to exponentially smaller)
- NFA/DFA are closed under complementation
  - build total DFA, swap accept/non-accept states
- For any regular language L, there is a unique minimal DFA that accepts L (up to isomorphism)
  - efficient algorithm to minimise DFA into equivalent DFA
  - partition refinement algorithm (like for bisimulation)
- Language emptiness of an NFA reduces to reachability
  - $L(A) \neq \emptyset$ iff can reach a state in F from an initial state in Q_0

Languages as properties

- Consider a model, i.e. an LTS/DTMC/MDP/…
  - e.g. DTMC $D = (S, s_{init}, \mathbf{P}, Lab)$
  - where labelling Lab uses atomic propositions from set AP
  - let $\omega \in Path(s)$ be some infinite path
- Temporal logic properties
  - for some temporal logic (path) formula ψ, does $\omega \vDash \psi$?
- Traces and languages
  - $trace(\omega) \in (2^{AP})^\omega$ denotes the projection of state labels of ω
  - i.e. $trace(s_0 s_1 s_2 s_3 \ldots) = Lab(s_0)Lab(s_1)Lab(s_2)Lab(s_3)\ldots$
  - for some language $L \subseteq (2^{AP})^\omega$, is $trace(\omega) \in L$?

Example



- Atomic propositions
  - $AP = \{ fail, try \}$
  - $2^{AP} = \{ \varnothing, \{fail\}, \{try\}, \{fail,try\} \}$
- Paths and traces
  - e.g. $\omega = s_0 s_1 s_1 s_2 s_0 s_1 s_2 s_0 s_1 s_3 s_3 s_3 \ldots$
  - $trace(\omega) = \varnothing\ \{try\}\ \{try\}\ \{fail\}\ \varnothing\ \{try\}\ \{fail\}\ \varnothing\ \{try\}\ \varnothing\ \varnothing\ \varnothing\ \ldots$
- Languages
  - e.g. "no failures"
  - $L = \{ \alpha_1 \alpha_2 \ldots \in (2^{AP})^\omega \mid \alpha_i \text{ is } \varnothing \text{ or } \{try\} \text{ for all } i \}$

Regular safety properties

- A safety property P is a language over 2^{AP} such that
  - for any word w that violates P (i.e. is not in the language), w has a prefix w', all extensions of which, also violate P
- A regular safety property is
  - safety property for which the set of "bad prefixes" (finite violations) forms a regular language
- Formally…
  - $P \subseteq (2^{AP})^\omega$ is a safety property if:
    - $\forall\ w \in ((2^{AP})^\omega \backslash P)\ .\ \exists$ finite prefix w' of w such that:
    - $P \cap \{\ w'' \in (2^{AP})^\omega \mid w' \text{ is a prefix of } w''\ \} = \varnothing$
  - P is a regular safety property if:
    - $\{\ w' \in (2^{AP})^* \mid \forall\ w'' \in (2^{AP})^\omega\ .\ w'.w'' \notin P\ \}$ is regular

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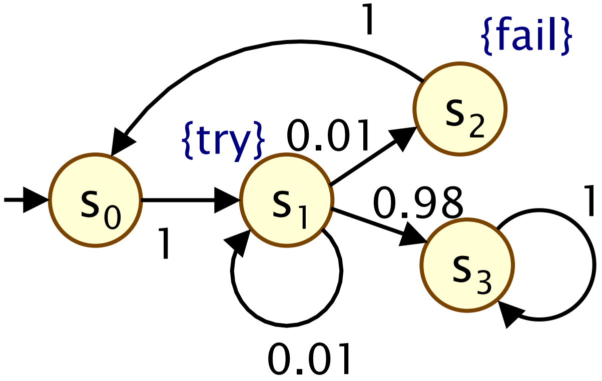
- Examples:
  - "at least one traffic light is always on"
  - "two traffic lights are never on simultaneously"
  - "a red light is always preceded immediately by an amber light"

Example

- Regular safety property:
  - "at most 2 failures occur"
  - language over:

    $2^{AP} = \{ \varnothing, \{fail\}, \{try\}, \{fail,try\} \}$



Example

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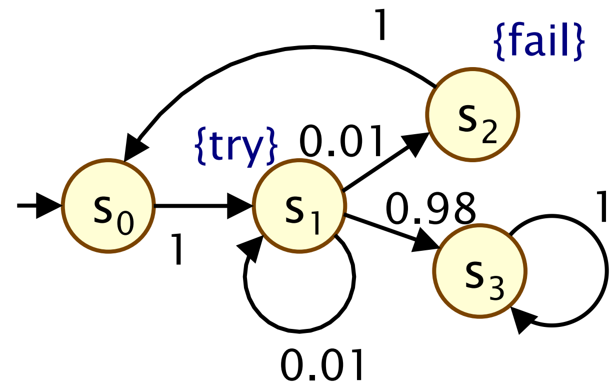
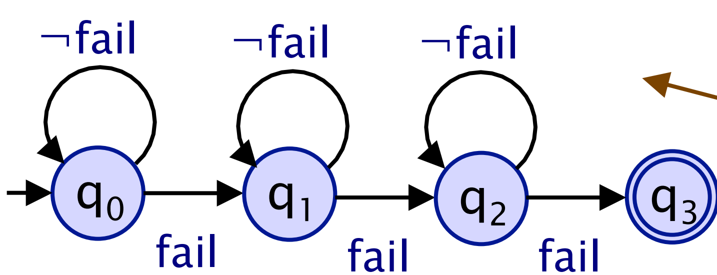


- Bad prefixes (regexp):

  (¬fail)*.fail.(¬fail)*.fail.(¬fail)*.fail

fail denotes:
({fail} + {fail,try})
¬fail denotes:
(∅ + {try})

- Bad prefixes (DFA):



fail denotes:
{fail}, {fail,try}
¬fail denotes:
∅, {try}

Regular safety properties + DTMCs

- Consider a DTMC D (with atomic propositions from AP) and a regular safety property $P \subseteq (2^{AP})^\omega$
- Let $\text{Prob}^D(s, P)$ denote the probability of P being satisfied
  - i.e. $\text{Prob}^D(s, P) = \text{Pr}^D_s\{ \omega \in \text{Path}(s) \mid \text{trace}(\omega) \in P \}$
  - where Pr^D_s is the probability measure over Path(s) for D
  - this set is always measurable (see later)
- Example (safety) specifications
  - "the probability that at most 2 failures occur is ≥ 0.999"
  - "what is the probability that at most 2 failures occur?"
- How to compute $\text{Prob}^D(s, P)$?

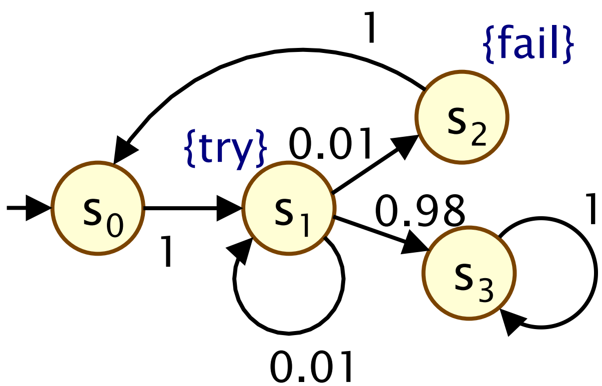
Product DTMC

- We construct the product of
  - a DTMC $D = (S, s_{init}, P, L)$
  - and a (total) DFA $A = (Q, \Sigma, \delta, q_0, F)$
  - intuitively: records state of A for path fragments of D

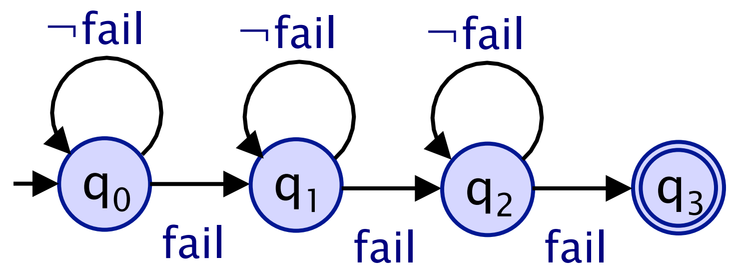
- The product DTMC $D \otimes A$ is:
  - the DTMC $(S \times Q, (s_{init}, q_{init}), P', L')$ where:
  - $q_{init} = \delta(q_0, L(s_{init}))$
  - $\mathbf{P}'((s_1,q_1),(s_2,q_2)) = \begin{cases} \mathbf{P}(s_1,s_2) & \text{if } q_2 = \delta(q_1, L(s_2)) \\ 0 & \text{otherwise} \end{cases}$
  - $L'(s,q) = \{ \text{accept} \}$ if $q \in F$ and $L'(s,q) = \varnothing$ otherwise

Example

DTMC D



DFA A

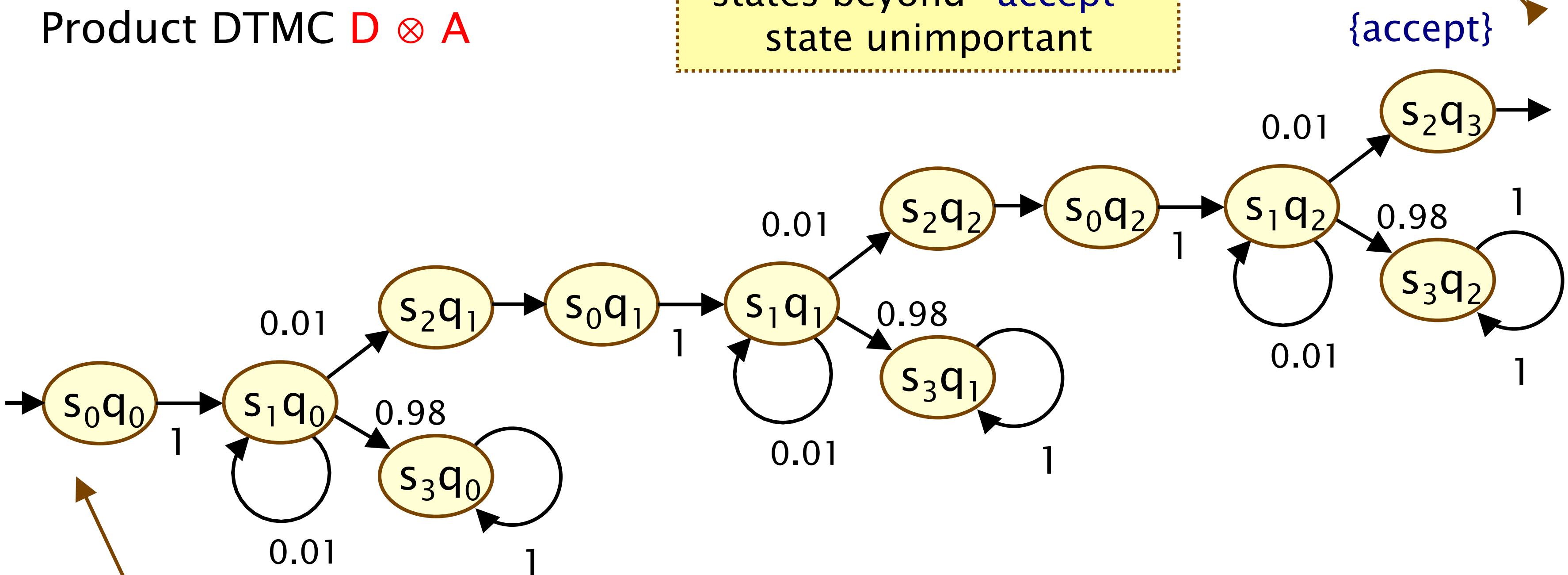


fail denotes:
{fail}, {fail,try}
¬fail denotes:
∅, {try}

Example

Product DTMC $D \otimes A$

states beyond "accept" state unimportant



$s_0,\delta(q_0,L(s_0))$

Product DTMC

- One interpretation of $D \otimes A$:
  - unfolding of D where q for each state (s,q) records state of automata A for path fragment so far
- In fact, since A is deterministic…
  - for any $\omega \in Path(s)$ of the DTMC D:
    - there is a unique run in A for trace(ω)
    - and a corresponding (unique) path through $D \otimes A$
  - for any path $\omega' \in Path^{D \otimes A}(s,q_{init})$ where $q_{init} = \delta(q_0,L(s))$
    - there is a corresponding path in D and a run in A
- DFA has no effect on probabilities
  - i.e. probabilities preserved in product DTMC

Regular safety properties + DTMCs

- Regular safety property $P \subseteq (2^{AP})^\omega$
  - "bad prefixes" (finite violations) represented by DFA A
- Probability of P being satisfied in state s of D
  - $\mathrm{Prob}^D(s, P) = \mathrm{Pr}^D_s\{ \omega \in \mathrm{Path}(s) \mid \mathrm{trace}(\omega) \in P \}$
    $= 1 - \mathrm{Pr}^D_s\{ \omega \in \mathrm{Path}(s) \mid \mathrm{trace}(\omega) \notin P \}$
    $= 1 - \mathrm{Pr}^D_s\{ \omega \in \mathrm{Path}(s) \mid \mathrm{pref}(\mathrm{trace}(\omega)) \cap L(A) \neq \varnothing \}$
  - where pref(w) = set of all finite prefixes of infinite word w

$$\boxed{\mathrm{Prob}^D(s, P) = 1 - \mathrm{Prob}^{D \otimes A}((s,q_s), \text{F accept})}$$

  - where $q_s = \delta(q_0, L(s))$

Example

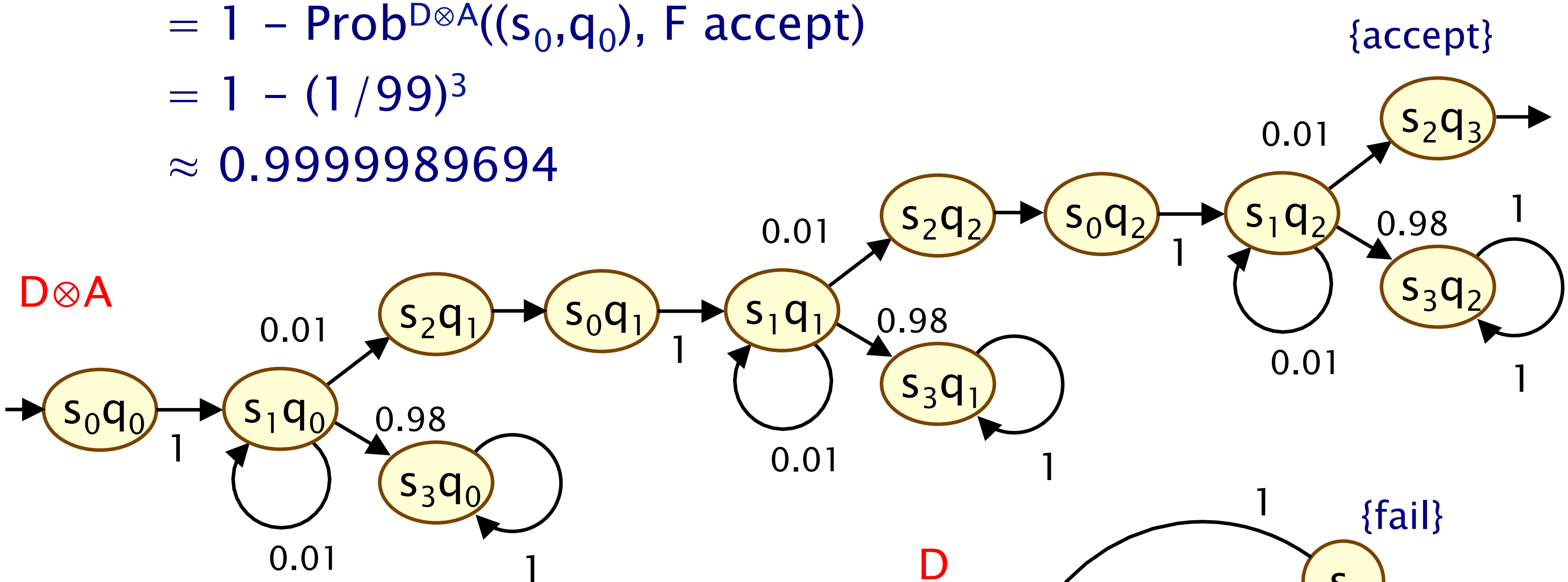
- $\mathrm{Prob}^{D}(s_0,$ "at most 2 failures occur")

  $= 1 - \mathrm{Prob}^{D\otimes A}((s_0,q_0), \text{F accept})$

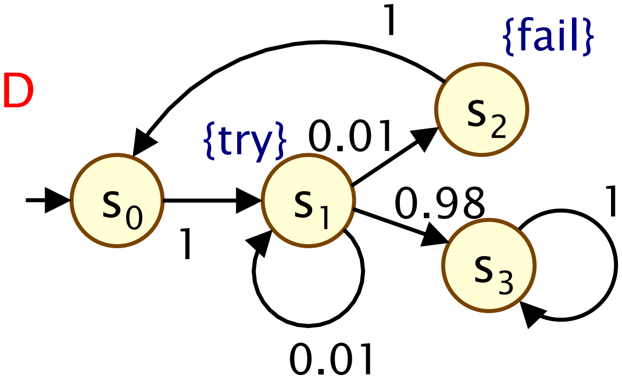
  $= 1 - (1/99)^3$

  ≈ 0.9999989694

D⊗A



D



Summing up…

- Nondeterministic finite automata (NFA)
  - can represent any regular language, regular expression
  - closed under complementation, intersection, …
  - (non-)emptiness reduces to reachability
- Deterministic finite automata (DFA)
  - can be constructed from NFA through determinisation
  - equally expressive as NFA, but may be larger
- Regular safety properties
  - language representing set of possible traces
  - bad (violating) prefixes form a regular language
- Probability of a regular safety property on a DTMC
  - construct product DTMC
  - reduces to probabilistic reachability